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An Approximation-Theoretic Characterization of Uniformly Rotund Spaces

INTRODUCTION

Let X be a Banach space and $M \subseteq X$ a closed subspace of X. We call M a Chebyshev subspace if for each $x \in X$ there exists a unique $\bar{m} \in M$ such that

$$\| x - \bar{m} \| = \text{dist}(x, M) = \inf\{\| x - m \| \mid m \in M\} = \| x + M \|;$$

in other words if each $x \in X$ has a unique best approximation in M, then, the map that associates with each $x \in X$ its unique best approximation in M is called the best approximation operator on M and is denoted by $P(M)$.

The Soviet mathematician A. L. Garkavi has said [3] "Every geometric question about Banach spaces has an equivalent expression as an approximation theoretic question." For example, each closed subspace is a Chebyshev subspace iff X is rotund and reflexive [2, 8]. In this note, we give conditions on best approximation operators equivalent to X being uniformly rotund (UR) and conditions equivalent to both X and X^* being UR (i.e., X being uniformly rotund and uniformly smooth). A less general version of this result appeared in [1] in connection with convergence of alternation sequences.

In general, $P(M)$ may be discontinuous. If X is UR, then, $P(M)$ is continuous, and in fact, Holmes [4] has shown that in this case, the class of maps $\{P(M) \mid M$ a closed subspace of $X\}$ is uniformly equicontinuous on bounded sets. We obtain Holmes' theorem as a corollary of our result. If $P(M)$ is actually uniformly continuous, then M is the range of a continuous linear projection [6], and so if $P(M)$ is uniformly continuous for each closed subspace, X is isomorphic to a Hilbert space [7].

A Banach space X, is rotund iff each norm-1 linear functional is tangent to the unit sphere of X at, at most, one point. The uniqueness of best approximation follows immediately from this definition. If the dual of X is rotund, then X is smooth, i.e., for each norm-1 vector $x \in X$, there is a unique norm-1 linear functional $f \in X^*$ with $f(x) = 1 = \| x \|$. Thus, for a smooth rotund space we have a well-defined norming map $n: S \to S^*$, where $n(x)(x) = 1$, and $n(x)(y) < 1$ for $y \neq x$. Here, S denotes the unit sphere of X and S^* the unit sphere of X^*.

If M is a closed subspace and $x \in X \backslash M$, then $m \in M$ is the best approximation to x iff $x - m \perp m'$ for all $m' \in M$. Here, $\perp$ denotes orthogonality in the sense of James [5], i.e., $x \perp y$ iff $\| x + ay \| \geq \| x \|$ for all real a. For a norm-1 vector in a smooth space this is equivalent to $n(x)(y) = 0$ because $n(x)$ is the Gateaux derivative of the norm functional at x. Thus, in smooth rotund reflexive spaces, $P(M)$ is characterized by the fact that $P(M)x \in M$ and $n(x - P(M)x) \in M^{\perp}$. Also, since M is a subspace, for any $m \in M$, $P(M)(x + m) = P(M)x + m$, and hence, $P(M)(x - P(M)x) = 0$.

There are several equivalent formulations of uniformly rotund. For our purposes, the following is convenient: X is UR iff for all sequences $\{x_i\}$, $\{y_i\}$ with $\| x_i \| = \| y_i \| = 1$, if $\| x_i + y_i \| \to 2$ then $\| x_i - y_i \| \to 0$.

2. A Characterization of UR Spaces

THEOREM 1. *Let X be a rotund reflexive Banach space. X is UR iff for every sequence $\{M_i\}$ of closed subspaces and every sequence $\{x_i\}$ with $\{\| x_i \|\}$ convergent such that for some $k > 0$,*

$$\| x_i \| \geq \| x_i - P(M_i)\, x_i \| \geq k, \qquad \textit{for all } i,$$

the following are equivalent:

(a) $\lim_i \| x_i \| = \lim_i \| x_i - P(M_i)\, x_i \|$,

(b) $\lim_i P(M_i)\, x_i = 0$.

If, in addition, X^* is rotund, then X^* is also UR iff the following are equivalent to (a) *and* (b) *for all sequences $\{x_i\}$ and $\{M_i\}$ satisfying the hypotheses.*

(c) $\lim_i \| x(x_i) - n(x_i - P(M_i)\, x_i)\| = 0$,

(d) $\lim_i \| n(x_i) - P(M_i^{\perp})\, n(x_i)\| = 0$,

(e) $\lim_i \| n(x_i)\,|_{M_i} \| = 0$,

(f) $\lim_i n(x_i)(m_i) = 0$ *uniformly on bounded sequences $\{m_i\}$ with $m_i \in M_i$ for each i.*

Proof. Suppose (a) and (b) are equivalent and $\{x_i\}$, $\{y_i\}$ are sequences with $\| x_i \| = \| y_i \| = 1$ and $\| x_i + y_i \| \to 2$. Let $M_i = \operatorname{span}\{x_i - y_i\}$. We shall show that $\lim_i P(M_i)\, x_i = \lim_i P(M_i)\, y_i = 0$ and so

$$\begin{aligned} y_i - x_i &= y_i - x_i + P(M_i)(x_i - P(M_i)\, x_i) \\ &= P(M_i)(y_i - x_i + x_i - P(M_i)\, x_i) \\ &= P(M_i)\, y_i - P(M_i)\, x_i \end{aligned}$$

also converges to 0. This gives UR in X.

Now, for each i, $\| x_i - P(M_i)\, x_i \| = \| x_i - \lambda_i(x_i - y_i)\|$ and $0 \leqslant \lambda_i \leqslant 1$. For, if $\lambda < 0$, then $\| x_i - \lambda(x_i - y_i)\| \geqslant (1 + |\lambda|) \| x_i \| - |\lambda| \| y_i \| = 1$, while if $\lambda = 1 + \mu$, with $\mu > 0$, then $\| x_i - (1 + \mu)(x_i - y_i)\| \geqslant (1 + \mu) \| y_i \| - \mu \| x_i \| = 1$.

However, if $0 \leqslant \lambda \leqslant 1$, then

$$\| x_i - \lambda(x_i - y_i)\| \leqslant (1 - \lambda) \| x_i \| + \lambda \| y_i \| \leqslant 1.$$

To show $P(M_i)\, x_i \to 0$, we need only prove that $\| x_i - \lambda_i(x_i - y_i)\| \to 1 = \| x_i \|$. If not, then for some $\delta > 0$, and some subsequence $\{x_k\}$, we have $\| x_k - \lambda_k(x_k - y_k)\| < 1 - \delta$. However, $\| x_k + y_k \| \to 2$, so for k large enough, $\| x_k + y_k \| > 2 - \delta$. Thus, $1 - \delta > \|(1 - \lambda_k)\, x_k + \lambda_k y_k \|$, and since $0 \leqslant \lambda_k \leqslant 1$, $1 \geqslant \| \lambda_k x_k + (1 - \lambda_k)\, y_k \|$. Adding, we have $2 - \delta > \| x_k + y_k \|$, a contradiction. Similarly, $P(M_i)\, y_i \to 0$, as required.

Now, assume that conditions (a)–(f) are equivalent, and X^* is also rotund. Let $\{f_i\}$, $\{g_i\}$ be sequences in X^* with $\|f_i\| = \|g_i\| = 1$ for all i, and $\|f_i + g_i\| \to 2$. Since X^* is now rotund and smooth, we may write $f_i = n(x_i)$ and $g_i = n(y_i)$, where $\| x_i \| = \| y_i \| = 1$.

Again, letting $M_i = \text{span}\{x_i - y_i\}$, we shall show that $P(M_i)\, x_i \to 0$, and $P(M_i)\, y_i \to 0$.

From condition (c),

$$\| n(x_i) - n(x_i - P(M_i)\, x_i)\| \to 0,$$

and

$$\| n(y_i) - n(y_i - P(M_i)\, y_i)\| \to 0.$$

However, $x_i - P(M_i)\, x_i = y_i - P(M_i)\, y_i$, and so

$$\begin{aligned} \|f_i - g_i\| &= \| n(x_i) - n(y_i)\| \\ &\leqslant \| n(x_i) - n(y_i - P(M_i)\, y_i)\| + \| n(y_i - P(M_i)\, y_i) - n(y_i)\| \\ &= \| n(x_i) - n(x_i - P(M_i)\, x_i)\| + \| n(y_i - P(M_i)\, y_i) - n(y_i)\|, \end{aligned}$$

which converges to zero proving UR in X^*.

Since $\| n(x_i) + n(y_i)\| \to 2$, for some sequence $\{z_i\}$ with $\| z_i \| = 1$, we have $n(x_i)(z_i) + n(y_i)(z_i) \to 2$. Hence, both $n(x_i)(z_i)$ and $n(y_i)(z_i)$ converge to 1, and so, $\lim_i n(x_i)(x_i - z_i) = \lim_i n(y_i)(y_i - z_i) = 0$.

If $\| x_i - z_i \|$ is bounded away from zero, then letting $N_i = \text{span}\{(x_i - z_i)\}$ and using condition (e), we have $P(N_i)\, x_i \to 0$. Therefore,

$$\begin{aligned} \left\| \frac{x_i + z_i}{2} \right\| &= \| x_i - \tfrac{1}{2}(x_i - z_i)\| \geqslant \| x_i - P(N_i)\, x_i \| \\ &\geqslant \| x_i \| - \| P(N_i)\, x_i \| \to 1. \end{aligned}$$

From the assumed equivalence of (a) and (b), X is (UR), and so $x_i - z_i \to 0$. Similarly, $y_i - z_i \to 0$, and so $x_i - y_i \to 0$. Since $P(M_i)\, x_i = \lambda_i(x_i - y_i)$, where $0 \leqslant \lambda_i \leqslant 1$, we have $P(M_i)\, x_i \to 0$, and likewise, $P(M_i)\, y_i \to 0$.

To show that the conditions are necessary, assume that X is UR and that $\{x_i\}$ and $\{M_i\}$ satisfy the hypotheses. First, note that the implication (b) $\Rightarrow$ (a) is obvious.

For (a) $\Rightarrow$ (b), we shall show that

$$\left\| \frac{x_i}{\| x_i \|} + \frac{x_i - P(M_i)\, x_i}{\| x_i - P(M_i)\, x_i \|} \right\| \to 2,$$

and so from UR in X,

$$\left\| \left(\frac{1}{\| x_i \|} - \frac{1}{\| x_i - P(M_i)\, x_i \|} \right) x_i + \frac{1}{\| x_i - P(M_i)\, x_i \|} P(M_i)\, x_i \right\| \to 0,$$

and hence, $P(M_i)\, x_i \to 0$.

To show that the sum converges to 2, we note that since $P(M_i)\, x_i$ is the best approximation to x_i in M_i,

$$\begin{aligned} &\left\| \left(\frac{1}{\| x_i \|} + \frac{1}{\| x_i - P(M_i)\, x_i \|} \right) x_i - \frac{1}{\| x_i - P(M_i)\, x_i \|} P(M_i)\, x_i \right\| \\ &\qquad \geqslant \left(\frac{1}{\| x_i \|} + \frac{1}{\| x_i - P(M_i)\, x_i \|} \right) \| x_i - P(M_i)\, x_i \| \to 2. \end{aligned}$$

Suppose now that X^* is also UR and (a) holds. Then,

$$\begin{aligned} \| n(x_i) + n(x_i - P(M_i)\, x_i) \| &\geqslant n(x_i) \left(\frac{x_i}{\| x_i \|} \right) + n(x_i - P(M_i)\, x_i) \left(\frac{x_i}{\| x_i \|} \right) \\ &= 1 + \frac{\| x_i - P(M_i)\, x_i \|}{\| x_i \|} \to 2, \end{aligned}$$

and (c) follows.

The implication (c) $\Rightarrow$ (d) is clear because $n(x_i - P(M_i)\, x_i) \in M_i^{\perp}$.

It is also obvious that (d) $\Rightarrow$ (e), because Phelps [8] has show that $\operatorname{dist}(n(x_i), M_i^{\perp}) = \| n(x_i) \mid_{M_i} \|$.

For (e) $\Rightarrow$ (f) for each i, we have

$$n(x_i)(m_i) = n(x_i)(m_i / \| m_i \|)\, \| m_i \| \leqslant \| n(x_i) |_{M_i} \| \, \| m_i \|.$$

Finally, assuming (f), we have

$$\begin{aligned} \| x_i - P(M_i)\, x_i \| &\geqslant n(x_i)(x_i) - n(x_i)(P(M_i)(x_i)) \\ &= \| x_i \| - n(x_i)(P(M_i)(x_i)), \end{aligned}$$

and since $\| P(M_i)\, x_i \| < 2 \| x_i \|$, condition (a) follows. Q.E.D.

COROLLARY (Holmes). *If X is UR and A is a bounded subset of X, then for each $\epsilon > 0$, there exists a $\delta(\epsilon, A) > 0$ such that for all closed subspaces $M \subseteq X$ and all $x, y \in A$, if $\| x - y \| < \delta$, then $\| P(M)x - P(M)y \| < \epsilon$.*

Proof. If the statement does not hold, then we must have bounded sequences $\{x_i\}$, $\{y_i\}$ and closed subspaces $\{M_i\}$ such that $\| x_i - y_i \| \to 0$ while

$$\| P(M_i)\, x_i - P(M_i)\, y_i \| \geqslant k > 0, \qquad \text{for all } i.$$

Let $z_i = x_i - P(M_i)\, y_i$, and $w_i = y_i - P(M_i)\, x_i$. From boundedness, we may assume that $\| z_i \|$, $\| z_i - P(M_i)\, z_i \|$, $\| w_i \|$, and $\| w_i - P(M_i)\, w_i \|$ all converge. Now,

$$\begin{aligned} P(M_i)\, z_i &= P(M_i)(x_i - P(M_i)\, y_i) \\ &= P(M_i)\, x_i - P(M_i)\, y_i , \end{aligned}$$

and we shall show that $P(M_i)\, z_i \to 0$.

If $\| z_i \| \to 0$, then, since $\| P(M_i)\, z_i \| \leqslant 2 \| z_i \|$, we are done. Otherwise, for $\epsilon > 0$ arbitrary and i large enough, we have

$$\begin{aligned} \| z_i \| + \epsilon &\geqslant \| z_i - P(M_i)\, z_i \| + \epsilon \\ &= \| x_i - P(M_i)\, y_i - P(M_i)(x_i - P(M_i)\, y_i) \| + \epsilon \\ &= \| x_i - P(M_i)\, x_i \| + \epsilon \\ &\geqslant \| x_i - P(M_i)\, x_i \| + \| y_i - x_i \| \\ &\geqslant \| y_i - P(M_i)\, x_i \| = \| w_i \|. \end{aligned}$$

Since the same argument can be repeated beginning with $\| w_i \| + \epsilon$, we may conclude that $\lim_i \| z_i \| = \lim_i \| z_i - P(M_i)\, z_i \| = \lim_i \| w_i \|$, and from Theorem 1, $P(M_i)\, z_i \to 0$. Q.E.D.

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